

## N O T I C E

THIS DOCUMENT HAS BEEN REPRODUCED FROM  
MICROFICHE. ALTHOUGH IT IS RECOGNIZED THAT  
CERTAIN PORTIONS ARE ILLEGIBLE, IT IS BEING RELEASED  
IN THE INTEREST OF MAKING AVAILABLE AS MUCH  
INFORMATION AS POSSIBLE

A77-11730

MEMORY EFFECTS IN TURBULENCE

J. O. Hinze

Translation of "Gedächtniseffekte in der Turbulenz,"  
Zeitschrift für angewandte Mathematic and Mechanik, Vol. 56, Oct. 1976,  
pp T 403 - T 415

(NASA-TM-75516) MEMORY EFFECTS IN  
TURBULENCE (National Aeronautics and Space  
Administration) 35 p HC A03/MF A01 CSCL 20D

N80-19451

Unclass  
G3/34 47442



NATIONAL AERONAUTICS AND SPACE ADMINISTRATION  
WASHINGTON, D.C. 20546 NOVEMBER 1979

## STANDARD TITLE PAGE

1. Report No. NASA TM-75516		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle MEMORY EFFECTS IN TURBULENCE				5. Report Date November 1979	
				6. Performing Organization Code	
7. Author(s) J. O. Hinze				8. Performing Organization Report No.	
				10. Work Unit No. NASW-3199	
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, California 94063				11. Contract or Grant No.	
				13. Type of Report and Period Covered Translation	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Washington, D.C. 20546				14. Sponsoring Agency Code	
15. Supplementary Notes  Translation of "Gedächtniseffekte in der Turbulenz," Zeitschrift für angewandte Mathematik und Mechanik, Vol. 56, Oct. 1976, pp T 403 - T 415 (A77-11730)					
16. Abstract  Memory effects in turbulent flow are studied. Based on experimental investigations of the wake flow of a hemisphere and cylinder, it is shown that such memory effects can be substantial and have a significant influence on momentum transport. Memory effects are described in terms of suitable memory functions.					
17. Key Words (Selected by Author(s))			18. Distribution Statement  Unclassified-Unlimited		
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 35	22. Price		

# List of Symbols

$$\bar{D}_{ij} = \frac{1}{2} [\partial \bar{U}_j / \partial x_i + \partial \bar{U}_i / \partial x_j]$$

$E_1$	one-dimensional spectral energy distribution
$G$	memory function
$k_1$	wave number in $x_1$ -direction
$L_1, L_t$	length scale
$L_0$	characteristic length
$P_t = \frac{1}{3} \rho \overline{u_j u_j}$	turbulent pressure
$p$	turbulent pressure fluctuation
$[Q_{2k}]_L$	Lagrangian autocorrelation
$q' = (q^2)^{1/2} = (\overline{u_1 u_1})^{1/2}$	
$R_L$	Lagrangian autocorrelation coefficient
$t$	time
$U$	characteristic velocity
[illeg.]	$x_1$ -component of mean velocity
[illeg.]	$x_1$ -component of turbulent velocity fluctuation
$u_1' = (u_1^2)^{1/2}$	intensity of $u_1$
$u^*$	shear stress velocity
$x$	coordinates
$y$	distance traveled in $x_1$ -direction
$\delta$	boundary layer thickness
$\delta_{ji}$	Kronecker delta
$\epsilon_m$	kinematic eddy viscosity
$\Lambda_1$	memory length scale
$\Lambda_{1f}$	axial integral length scale
$\Lambda_{1L}$	Lagrangian integral length scale
$\nu$	kinematic viscosity
$\Omega$	intermittence factor
$\rho$	density
$T_{1L}$	Lagrangian integral time scale



## MEMORY EFFECTS IN TURBULENCE

J. O. Hinze

It is shown that the effects in a turbulent flow are not strictly local and that memory effects play an essential role. Thus, momentum-transfer processes are not determined by the local gradient of the mean velocity alone. /403\*

The Reynolds stresses can be thought of as consisting of two parts. The first part is the stress that would occur if it were determined solely by the local state of the flow, expressed in the local mean velocity gradient. The second part is an additional contribution from a memory of the influence of a change in this local gradient in the memory region of interest.

A transfer of the gradient type will often occur in fully developed flows, because either the additional memory effects are relatively small, or the flow is self-preserving.

These additional memory effects can be directly detected in the development regions of turbulent flows, where the turbulence has no opportunity to adapt to new local flow conditions due to the rapid fluctuation of the main flow direction.

The memory effect can be described with a suitable memory function. As an approximation, a simple exponential function is assumed for this. In this case the Reynolds stress is subject to a relaxation equation whose solution yields the axial distribution of the stress.

---

\* Numbers in the margin indicate pagination in the foreign text.

Based on experimental studies of the wake flow behind a hemisphere in a turbulent boundary layer as well as the wake flow of a cylinder, it is shown that such additional memory effects can be substantial. Both cases involve a flow in the development region which fluctuates rapidly in the main flow direction. If the shear stress under consideration is expressed by the velocity gradient and an eddy viscosity, this eddy viscosity is equal to the value of the fully developed flow after correction for the additional memory effect.

## 1. General

Flows which exhibit memory effects have been known for some time, mainly for non-Newtonian fluids. In this case the fluid should at least have viscoelastic properties which, as in rheology, are determined by its molecular structure.

But memory effects are also present in the turbulent flow of Newtonian fluids. The phenomenological theories of Prandtl and Taylor on the momentum transfer associated with turbulent or Reynolds shear stress essentially involve a memory effect determined by the Lagrangian autocorrelation of fluid particles in turbulent motion. Subsequent studies have shown that various phenomena in turbulence can be explained in principle by the assumption of a viscoelastic turbulence behavior, as in the formation of secondary flows in turbulent flow through a pipe of noncircular cross-section. It is apparent, then, that turbulence may be regarded as a hypothetical non-Newtonian fluid. A basic departure from rheology must be made, however.

First, the principle of local action is basic to rheology. According to this principle, the stresses in a particle are determined or influenced by motions of the medium outside a certain very small material environment of the particle. "Particle" here refers to a small quantity of matter with a specific identity and a

volume which is very small in relation to the macroscopic length /404  
scale of the motions. The state of stress at a particular point in time is determined entirely by the history of the motion in the immediate environment of the particle up to that time.

For "rheologically simple" fluids, the deformation history is described completely by the local deformation gradient (velocity gradient). Simple fluids are isotropic, and they "remember" the past only via the deformation tensor up to the point of time under consideration.

Second, a constitutive equation, i.e., a relation between deformation and stress, must satisfy various conditions. The equation must be invariant with respect to a transformation of the coordinate system. It should also be invariant with respect to time-dependent rigid motions such as translation and rotation; in other words, it should be objective.

A century ago, Boussinesq regarded turbulent flow as a hypothetical fluid, specifically a Newtonian fluid. He assumed a linear relation between the stress tensor and the deformation tensor by using an effective "eddy viscosity" as a scalar quantity. With a small extension, Boussinesq's relation can be written

$$-\overline{u_i u_i} = -\frac{1}{\rho} \bar{P}_t \cdot \delta_{ii} + \epsilon_m \left( \frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i} \right). \quad (1)$$

where  $\bar{P}_t = \frac{1}{3} \rho \overline{u_j u_j}$  is the mechanical pressure, and  $\epsilon_m$  is the kinematic eddy viscosity.

Because the turbulent velocity fluctuations  $u_i$  are difference velocities and thus objective, as is the deformation tensor, relation (1) is objective.

If turbulence is regarded as a rheologically simple fluid with non-Newtonian properties, relation (1) can be further supplemented as follows:

$$-\overline{u_i u_i} = F_n(x_k, t, \bar{U}_k, \bar{P}_i/\rho, \partial \bar{U}_k / \partial x_m). \quad (2)$$

It can be shown that if the objectivity conditions are met, the following relation can be derived [1]:

$$-\overline{u_i u_i} = \epsilon_0 \delta_{ii} + 2\epsilon_m \bar{D}_{ii} + 4\epsilon_c \bar{D}_{ik} \bar{D}_{ki} \quad (3)$$

where

$$\bar{D}_{ii} = 1/2 (\partial \bar{U}_i / \partial x_i + \partial \bar{U}_i / \partial x_i).$$

The quantities  $\epsilon_0$ ,  $\epsilon_m$  and  $\epsilon_c$  are still functions of the principal invariants of  $\bar{D}_{ij}$  and of the possible scalar invariants such as turbulent pressure. This nonlinear relation is analogous to the rheological equation of state for viscous fluids of Stokes and of Reiner-Rivlin. This relation satisfies the condition of local action but contains no "memory."

For a local action in turbulent flows, one must also consider very small particles which do not exceed the Kolmogoroff micro length scale in order of magnitude. Now it happens that such small particles lose their identity (momentum) too quickly to make a significant contribution to turbulent transport. To be effective in this regard, the lumps of fluid must be rather large -- of the same order of magnitude as the energy-containing eddies, i.e., the order of magnitude of the integral length scale. The action is essentially no longer local, and so higher derivatives of the mean velocity must also be considered. The radius of action is taken to be proportional to the eddy size under consideration. In this case the size of a unit area for momentum

transport ( $\overline{u_j u_1}$ ) can no longer be taken as small, but must be of the order of magnitude of these lumps of fluid.

Memory effects in turbulence, expressed in a memory or relaxation distance, are determined in part by the size of the eddies. A classic example of the dependence of the relaxation or memory distance on eddy size is found in the studies of Clauser [2]. Fig. 1 shows the result of these studies. A rod ( $d = 12.5$  mm) is placed in a turbulent boundary layer of thickness  $\delta = 235$  mm in a cross-stream direction, thereby producing a flow which is characterized by the local velocity defect  $\Delta \bar{U}_1$ . The decay of this defect occurs much more rapidly near the wall ( $x_2/\delta = 0.16$ ; smaller eddies) than at a point more distant from the wall ( $x_2/\delta = 0.59$ ; larger eddies).

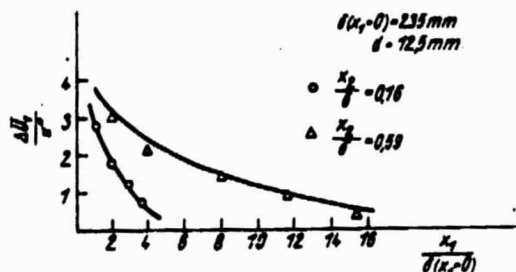


Fig. 1. Decay of a mean-velocity disturbance in a turbulent boundary layer.

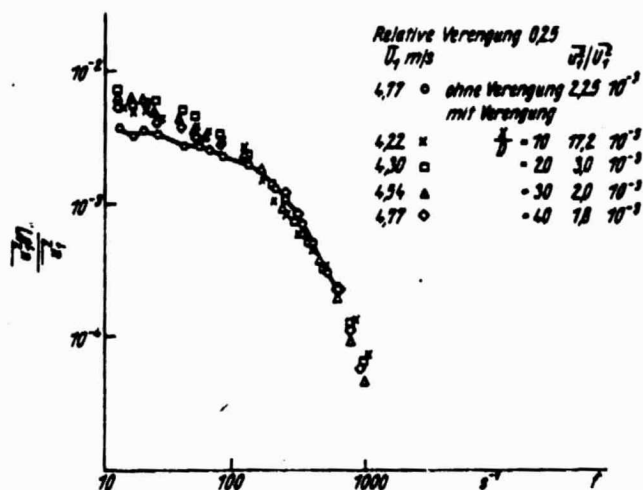


Fig. 2. Spectral energy distribution of turbulence along a pipe axis.  
Key: a - Relative constriction 0.25; b - Without constriction; c - With constriction.

In his wave theory of turbulence, Landahl [3] showed that the only suitable length scale for the spatial behavior of an eddy is the size of the eddy itself. A wave component loses its identity at a distance of approximately six of its own wavelengths. Measurements in a boundary layer showed that at  $x_2/\delta = 0.5$ , the Lagrangian integral length scale is approximately  $4\delta$ . Thus, the disturbance will decay completely only after about  $24\delta$ . The dependence of the decay of velocity disturbances on eddy size was nicely confirmed by the experimental results of Lissenburg [4]. The turbulent flow of air in a straight pipe of circular cross-section ( $D = 20$  mm) was disturbed by a local constriction of the pipe. At various distances downstream of the constriction, measurements were made of mean velocity distribution and the intensity of turbulent velocity along the pipe axis and close to the wall. The Reynolds number relative to diameter and mean velocity was approximately 5000.

The results of this experiment are shown in Fig. 2. After  $x = 20 D$  downstream of the constriction, the spectral energy distribution is already equal to the undisturbed flow for frequencies higher than  $100 \text{ s}^{-1}$ . For lower frequencies, the spectral distribution is not yet equal to the undisturbed flow even after  $40 D$ . If, as an approximation, the propagation velocity of the disturbances is assumed to equal the mean velocity, Landahl's estimate for  $20 D (= 0.4 \text{ m})$  leads to a frequency of  $70 \text{ s}^{-1}$ , which is of the same order of magnitude as  $100 \text{ s}^{-1}$ . /405

It can be concluded from Landahl's work that the more the transfer of size is governed by the fine structure of the turbulence, the more local is the effect. This validates the assumption of a transfer of gradient type which is made in many new turbulence theories in order to determine the influence of velocity correlations of third and higher order.

Because the larger eddies are effective in momentum transfer, with an action that is not limited to small regions, it may be asked whether the requirements of objectivity, and especially the condition of invariance under rotation, should still be satisfied, because centrifugal and Coriolis effects can no longer be neglected. Therefore, Lumley [1] has suggested that a constitutive equation for turbulence as a hypothetical fluid no longer be subjected to this last condition.

For the momentum transfer  $u_1$ , we may write the following general expression for the time-averaged transport through a unit of area at right angles to the  $x_2$ -direction:

$$\overline{u_1 u_1} = \frac{1}{T} \int_0^T dt_0 u_1(t_0) U_1(t_0). \quad (4)$$

The value  $\bar{U}_1(t_0)$  is determined by the prior history of the particle for all  $t \leq t_0$ . This prior history is determined in part by the distribution of mean velocity in the entire region under consideration and by the scale of the interaction between the particle and its environment during their respective motions. Averaged over many particles, this is associated with the Lagrangian autocorrelation. In principle, this aftereffect can be expressed in terms of a suitable memory or relaxation function. But the determination of such a suitable function generally presents a difficulty which has not yet been overcome.

For a particle which has been in motion for a time  $t$  and has traveled a distance  $y_k$ , a series expansion of  $U_1(t_0; t)$  yields

$$U_1(t_0; t) = (\bar{U}_1)_0 - y_1(t_0; t) \left( \frac{\partial \bar{U}_1}{\partial x_1} \right)_0 + \frac{1}{2} y_1 y_1(t_0; t) \left( \frac{\partial^2 \bar{U}_1}{\partial x_1 \partial x_1} \right)_0 - \frac{1}{6} y_1 y_1 y_1(t_0; t) \left( \frac{\partial^3 \bar{U}_1}{\partial x_1 \partial x_1 \partial x_1} \right)_0 + \dots \quad (5)$$

If the first derivative of the mean velocity is sufficient, relation (4), with

$$y_2(t_0; t) = \int_0^t dt' u_2(t_0 - t'),$$

yields

$$-\overline{u_1 u_1} = \frac{\partial \bar{U}_1}{\partial x_1} \int_0^t dt' \overline{u_2(t_0) u_2(t_0 - t')} = \frac{\partial \bar{U}_1}{\partial x_1} \int_0^t dt' [Q_{22}]_L(t') = \epsilon_{22}(t) \frac{\partial \bar{U}_1}{\partial x_1}; \quad (6)$$

$[Q_{2k}]_L = u_2(t_0) u_k(t_0 - t)$  is the Lagrangian autocorrelation, and  $\epsilon_{2k}$  is an eddy viscosity which can have various values for the various directions  $x_k$ .

/406

For  $t \rightarrow \infty$ , we obtain the relation of Boussinesq, but with various possible values of the eddy viscosity  $\epsilon_{2k}$ . The relation is not objective. Since the relation contains only the first velocity derivative, it appears as if the action is local in character. However, it must be noted that the appearance of the Lagrangian autocorrelation points to an influence of prior history. This influence has been nicely illustrated by Philip [5]. He imagines the velocity of a particle at each point in time as consisting of two parts. The first part corresponds to that which the particle still "remembers" of its velocity  $u(0)$  at time  $t = 0$ , while the second part consists of the momentum contributed by the exchange with its momentary environment:

$$u(t) = u(0) R_L(t) + u^*(t).$$

$R_L(t)$  is the Lagrangian autocorrelation coefficient and describes the influence of the past.

For a stationary homogenous turbulence, it should be true that



$u^2(t) = \overline{u^2}(0) = \text{const.}$  Philip calls the first part  $u_1(t) = u(0) R_L(t)$  the drift velocity. For very large times  $t$ , the memory of the original velocity becomes vanishingly small.

It is assumed in the simple relation (6) that the velocity gradient in the region through which a particle passes during the memory period is sufficiently constant. In most turbulent flows of interest, however, this is not the case. Corrsin [6] recently made a critical study of the conditions which must be satisfied so that the assumption of a turbulent diffusion of the gradient type may be made. It is shown that of the various conditions, the most important is that the length scale of the transport mechanism should be small relative to the distance over which the curvature of  $\partial \overline{U}_1 / \partial x_2$ , i.e.  $\partial^2 \overline{U}_1 / \partial x_2^2$ , undergoes a substantial change. Corrsin considered a one-dimensional diffusion process in the  $x_2$ -direction and a spatially symmetrical transport mechanism, so that the term with the second derivative is absent in a series expansion such as eqn. (5). Experience has shown that the error in assuming a diffusion of the gradient type is not large. This is due to the fact that the diffusion is essentially one-dimensional in most cases.

In the cases of three-dimensional transport considered here, the terms with the second derivative must also be considered in series expansion (6) as a second approximation, without assuming a diffusion of the gradient type. If this is not done, a non-negligible error can be made locally in the distribution of the mean velocity. As early as 1942, Prandtl [7] pointed this out in a modification of his mixing-length theory for the environment of the maximum velocity of a free jet, in which the first velocity derivative vanishes. In this modification he proposed a statistical mean for the first and second derivatives.

Based on his measurements of turbulent diffusion of the

kinetic energy of the three turbulent velocity components in the wake flow of a cylinder, Townsend [8] concluded that there are places in the wake where diffusion takes place against the gradient -- specifically, near the energy-distribution maximum of the turbulence in a cross-section of the wake flow.

It is also known that near a mean-velocity maximum in the case of asymmetrical flows, the turbulent shear stress and the mean-velocity gradient may have opposite signs, which can be interpreted as a "negative energy production" of the turbulence [9-13]. The explanations advanced by Beguier [12], Hinze [14], Eskinazi [15] and other authors are all based on a non-negligible influence of large eddies with a memory effect that requires at least that the second derivative of the mean velocity be taken into account.

Apparently the turbulent shear stress, or more generally, the Reynolds stresses, cannot be regarded as a local property of the flow. This led Philips [16] to suggest that one should regard the gradient of the Reynolds stress, rather than the Reynolds stress itself, as a local property.

In the cases previously mentioned, one must take into account the second velocity derivative, i.e., the change in the velocity gradient perpendicular to the main flow. But there are also cases in which the change of this gradient in the main flow occurs relatively quickly with a time or length scale which is no longer large in comparison with the scale of the large energy-containing eddies. In this respect the turbulence is no longer "adapted" to or "in equilibrium" with the local conditions of the flow.

In his study of boundary layers which undergo rapid spatial changes in the main-flow direction, Deissler [17] showed that the Reynolds stress remains approximately constant along one streamline

and may be equated with the initial value. In these necessarily short boundary layers, the memory distance extends over the entire length of the boundary layer considered.

Other examples in which memory effects of the type mentioned have been detected are the wake flow behind a hemisphere in a turbulent boundary layer and the wake flow immediately behind a cylinder. These two cases will be discussed in greater detail below.

The memory effects can be allowed for if the series expansion of  $\bar{U}_1$  according to eqn. (5) is continued in relation (4), and first and higher derivatives of  $\partial \bar{U}_1 / \partial x_k$  are also considered. Relation (4) then becomes

/407

$$-\overline{u_2 u_1} = \int_0^t dt' \left[ \overline{u_2(t_0) u_2(t_0 - t')} - \frac{1}{2} \overline{u_2(t_0) u_2(t_0 - t') y_i(t_0; t')} \frac{\partial}{\partial x_i} + \right. \\ \left. + \frac{1}{6} \overline{u_2(t_0) u_2(t_0 - t') y_i y_m(t_0; t')} \frac{\partial^2}{\partial x_i \partial x_m} + \dots \right] \frac{\partial \bar{U}_1}{\partial x_k}. \quad (7)$$

In most cases it is sufficient to take into account only the terms up to the second derivative in a series expansion as a second approximation. In a manner analogous to relation (6), we find from eqn. (7)

$$-\overline{u_2 u_1} = \varepsilon_{2k}(t) \frac{\partial \bar{U}_1}{\partial x_k} - \frac{1}{2} \int_0^t dt' \overline{u_2(t_0) u_2(t_0 - t') y_i(t_0; t')} \frac{\partial^2 \bar{U}_1}{\partial x_i \partial x_k} \\ = \varepsilon_{2k}(t) \frac{\partial \bar{U}_1}{\partial x_k} - \frac{1}{2} \varepsilon_{2k}^0(t) L_i(t) \frac{\partial^2 \bar{U}_1}{\partial x_i \partial x_k}.$$

The last term in an approximation based on the assumption that the integral of a triple correlation can be written as the product of a suitable length  $L_1(t)$  and an eddy viscosity. A further approximation is that a scalar eddy viscosity is sufficient for  $\epsilon_{2k}$  and  $\epsilon_{2k}^0$ , because it is also assumed that  $t \rightarrow \infty$ :

$$-\overline{u_1 u_1} = \left[ \epsilon_m \frac{\partial \bar{U}_1}{\partial x_2} - \frac{1}{2} \epsilon_m^0 L_1 \frac{\partial^2 \bar{U}_1}{\partial x_1 \partial x_2} \right]. \quad (8)$$

This equation can be generalized to a relation for  $-\overline{u_j u_i}$ , one which is symmetrical in  $j$  and  $i$ . Because  $\partial \bar{U}_1 / \partial x_1 = 0$  for an incompressible medium, we write the following expression so that the relation for  $\overline{u_j u_i}$  contains no contradiction:

$$-\overline{u_j u_i} = -\frac{\bar{P}_t}{\rho} \delta_{ji} + 2 \left[ \epsilon_m \bar{D}_{ji} - \frac{1}{2} \epsilon_m^0 L_1 \frac{\partial}{\partial x_i} \bar{D}_{ji} \right]. \quad (9)$$

The second velocity derivative in eqn. (8) and eqn. (9) may signify a nonlocal action. This depends on what the length scale  $L_1$  is determined by. At right angles to the mean-flow direction, inclusion of this term signifies a nonlocal action. In the main-flow direction, however,  $L_1$  can become relatively large due to the high convection velocity, and the action may be local in character.

The influence of a spatial change in  $\bar{D}_{ji}$  on momentum transfer can also be taken into account with a memory function. Instead of eqn. (9) we then consider

$$-\overline{u_j u_i} = -\frac{\bar{P}_t}{\rho} \delta_{ji} + 2 \int_0^\infty dt' \epsilon_m(t') \bar{D}_{ji}(t - t') G(t'). \quad (10)$$

For large times  $t$ , the normalized memory function  $G(t)$  decays to zero. It is characteristic of the turbulent flow under consideration. Eqn. (10) points essentially to a local action. As mentioned earlier, however, relatively large lumps of fluid play a role in momentum transfer. If  $\bar{D}_{ji}$  varies noticeably in the range of the order of magnitude of these lumps, the action is not local. However, a description based on eqn. (10) may be possible if the transfer takes place as if the action were local.

Relation (10) is similar to that for a viscoelastic fluid. Crow [18] has suggested such a relation for describing the properties of the fine structure of a turbulent flow.

## 2. Theory

In this section we shall examine relations (9) and (10) more closely. Of particular interest for the cases investigated experimentally is the shear stress per unit mass ( $-\overline{u_2 u_1}$ ), because these cases involve nearly one-dimensional flows with a pronounced main-flow direction. The mean flow is two-dimensional with the velocity components  $\bar{U}_1$  in the main-flow direction  $x_1$ , and  $\bar{U}_2$  in the direction  $x_2$  at right angles to the main-flow direction, where  $\bar{U}_1 \gg \bar{U}_2$ . The mean flow is independent of the  $x_3$ -direction.

As a second approximation, it is assumed that the memory effect of  $\partial \bar{U}_1 / \partial x_2$ , i.e., the influence of a relatively rapid change of  $\partial \bar{U}_1 / \partial x_2$ , is sufficient in the main-flow direction. According to eqn. (9),

$$-\overline{u_1 u_2} = \left[ e_m \frac{\partial \bar{U}_1}{\partial x_2} - \Lambda_1 e_m^0 \frac{\partial^2 \bar{U}_1}{\partial x_1 \partial x_2} \right] \quad (11)$$

with  $\Lambda_1 = 1/2 L_1$ .

Analogously, eqn. (10) gives

/408

$$-\overline{u_2 u_1}(t) = \int_0^{\infty} dt' \epsilon_m(t') \frac{\partial \bar{U}_1}{\partial x_2}(t-t') G(t'). \quad (12)$$

The last expression is more general than eqn. (11). The significance of these relations is that the shear stress may be envisioned as consisting of two parts, hence:

$$-\overline{u_2 u_1} = (-\overline{u_2 u_1})_g + (-\overline{u_2 u_1})_e.$$

The first part  $(-\overline{u_2 u_1})_g$  is the shear stress which would occur if it were determined entirely by the "local" state of the flow. Here "local" does not necessarily refer to a small region. It may be said that this part is "in equilibrium" with the local state. It may, for example, be equated with the shear stress according to the Boussinesq relation  $\epsilon_m \partial \bar{U}_1 / \partial x_2$  if the memory remains confined to a region in which  $\partial \bar{U}_1 / \partial x_2$  is practically constant.

The second part  $(-\overline{u_2 u_1})_e$  is an additional contribution which a lump of fluid "remembers" during the transfer process from the influence of a variable  $\partial \bar{U}_1 / \partial x_2$  in the memory region considered.

Usually the relative intensity of the turbulence is not large, and it may be assumed that Taylor's hypothesis of a frozen turbulence is valid:  $x_1 \approx \bar{U}_1 t$ . Eqn. (12) can then be written in a Eulerian description:

$$-\overline{u_2 u_1}(x_1) = \int_{-\infty}^{x_1} dx'_1 \epsilon_m(x') \frac{\partial \bar{U}_1}{\partial x_2}(x'_1) G(x_1 - x'_1). \quad (13)$$

This expression contains the contributions  $(-\overline{u_2 u_1})_g$  and  $(-\overline{u_2 u_1})_e$ . If  $\epsilon_m \partial \overline{U_1} / \partial x_2 = \text{const.}$  in the region in which  $G(x_1 - x'_1) > 0$ , we obtain with  $\int_{-\infty}^{x_1} dx'_1 G(x_1 - x'_1) = 1$  the Boussinesq relation, and  $(-\overline{u_2 u_1})_e = 0$ .

As mentioned, eqn. (13) implies essentially a local action. In the present work this relation is successfully applied to several turbulent flows. In these cases it is apparently correct to assume that the momentum transfer is local in character but has a memory.

An obvious step is to associate the memory function with a Lagrangian autocorrelation. Although in principle the Lagrangian autocorrelation is not equal to a simple exponential function, such an assumption is sometimes made with satisfactory results in order to simplify calculations. We shall make the same assumption here for the memory function; thus, in normalized form, we obtain

$$G(x_1 - x'_1) = \frac{1}{\Lambda_1} \exp\left(-\frac{x_1 - x'_1}{\Lambda_1}\right); \quad (14)$$

$\Lambda_1$  is then an effective memory length. With this expression for  $G$ , eqn. (13) becomes

$$-\overline{u_2 u_1}(x_1) = \frac{1}{\Lambda_1} \int_{-\infty}^{x_1} dx'_1 \epsilon_m(x'_1) \frac{\partial \overline{U_1}}{\partial x_2}(x'_1) \exp\left(-\frac{x_1 - x'_1}{\Lambda_1}\right). \quad (15)$$

Owing to the exponential function, expression (15) can now be regarded as the solution of an inhomogenous differential equation of first order, which is compatible with the relaxation process [19] considered above. This differential equation is

$$\Lambda_1 \frac{\partial}{\partial x_1} (-\overline{u_2 u_1}) + (-\overline{u_2 u_1}) = \epsilon_m \frac{\partial \overline{U_1}}{\partial x_2}. \quad (16)$$

Up to now we have assumed the length  $\Lambda_1$  to be independent of  $x_1$ . However, experiments have shown that  $\Lambda_1$  undergoes noticeable changes over distances equal to its length. Equation (16) can be expanded in the sense that  $\Lambda_1$  may still be a function of  $x_1$ . The solution is then

$$-\overline{u_1 u_1}(x_1) = -\overline{u_1 u_1}(x_{10}) \exp \left[ - \int_{x_{10}}^{x_1} \frac{dx'_1}{A_1(x'_1)} \right] + \int_{x_{10}}^{x_1} dx'_1 \varepsilon_m(x'_1) \frac{\partial \bar{U}_1}{\partial x_1}(x'_1) \frac{1}{A_1(x'_1)} \exp \left[ - \int_{x'_1}^{x_1} \frac{dx''_1}{A_1(x''_1)} \right]. \quad (17)$$

The integrals in the exponential expressions can be evaluated if we assume a power function for  $\Lambda_1$ :

$$A_1(x_1) = ax_1^n. \quad (18)$$

Solution (17) for  $n = 1$  then becomes

/409

$$\cdot \frac{1}{1-\nu/l_x} \cdot (l_x) \frac{x_0}{1/l_x} (l_x)^{n-1} \exp \int_{l_x}^{x_0} \frac{\nu}{\nu/l_x - x} + \frac{(0l_x)}{\nu/l_x} \cdot (0l_x) \underline{l_n^{n-1}} - = (l_x) \underline{l_n^{n-1}} - \quad (19)$$

For  $n \neq 1$  we obtain

$$\begin{aligned}
 -\overline{u_2 u_1}(x_1) = & -\overline{u_2 u_1}(x_{10}) \exp \left[ -\frac{x_1^{1-n} - x_{10}^{1-n}}{a(1-n)} \right] + \\
 & + \frac{1}{a} \exp \left[ -\frac{x_1^{1-n}}{a(1-n)} \right] \cdot \int_{x_0}^{x_1} dx'_1 \epsilon_m(x'_1) \frac{\partial \bar{U}_1}{\partial x_2}(x'_1) x_1'^{-n} \exp \left[ \frac{x_1'^{1-n}}{a(1-n)} \right].
 \end{aligned} \quad (20)$$

An equation for the term  $\partial/\partial x_1(-\overline{u_2 u_1})$ , which occurs in eqn. (16), is also derived from the dynamic equation for the Reynolds stress, specifically [20]:



$$\bar{U}_1 \frac{\partial}{\partial x_1} (-\overline{u_1 u_1}) = \overline{u_1^2} \frac{\partial \bar{U}_1}{\partial x_1} + \frac{\partial}{\partial x_1} \overline{u_1 \left( u_1^2 + \frac{p}{\rho} \right)} - \frac{p}{\rho} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) + \nu \left[ 2 \frac{\partial u_1}{\partial x_1} \frac{\partial u_1}{\partial x_1} + \frac{\partial^2}{\partial x_1^2} (-\overline{u_1 u_1}) \right].$$

The second term on the right side is usually negligibly small.  
Disregarding viscosity effects, the equation reduces to

$$\bar{U}_1 \frac{\partial}{\partial x_1} (-\overline{u_1 u_1}) \simeq \overline{u_1^2} \frac{\partial \bar{U}_1}{\partial x_1} - \frac{p}{\rho} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right).$$

For the pressure term, Rotta [21] suggested the following combination with the shear stress:

$$-\frac{p}{\rho} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) = -A \frac{q'}{L} (-\overline{u_1 u_1}),$$

where  $q' = (\overline{q^2})^{1/2} = (\overline{u_1 u_1})^{1/2}$ ;  $L$  is a length scale of the size of an integral length scale; and  $A$  is a numerical constant. Hence,

$$\frac{\partial}{\partial x_1} (-\overline{u_1 u_1}) \simeq \frac{\overline{u_1^2}}{\bar{U}_1} \frac{\partial \bar{U}_1}{\partial x_1} - \frac{A}{L} \frac{q'}{\bar{U}_1} (-\overline{u_1 u_1}), \quad (21)$$

$L$  is, as mentioned, of the size of the integral length scale. For this we select  $\Lambda_{1f}$  as an axial integral length scale and set  $L = \text{const } \Lambda_{1f}$ . We also set  $A_1 = \bar{U}_1 I_{1L} = (\bar{U}_1 / u'_1) A_{1L} = \text{const } \bar{U}_1 / u'_1 A_{1L}$  with  $u'_1 = (\overline{u_1^2})^{1/2}$ ;  $I_{1L}$  is the Lagrangian integral time scale,  $\Lambda_{1L}$  is the Lagrangian integral length scale. If now  $\Lambda / L \cdot q' / \bar{U}_1 = \text{const} \cdot q' / u'_1 \cdot 1 / \Lambda_1$ , we obtain eqn. (16) from eqn. (21) if  $\text{const } q' / u'_1 = 1$ . Further let  $\overline{u_1^2} \Lambda_1 / \bar{U}_1 = \overline{u_1^2} I_{1L} = \epsilon_m I_{1L} / I_{2L}$ , and for the Lagrangian integral time scale,  $I_{1L} \sim I_{1L}$ , which is the case only in an isotropic state of turbulence and is true only in rough approximation for a nonisotropic turbulence. In any case, the relaxation

equation (16) is not incompatible with the dynamic equation for shear stress.

It is our intention to show memory effects in flows which fluctuate rapidly in the mean flow direction. It may be asked whether a memory effect can be present in flows which do not fluctuate rapidly in this way. According to relations (11) and (16), the memory effects are negligible if

$$A_1 \delta^2 \bar{U}_1 / \partial x_1 \partial x_1 \ll \delta \bar{U}_1 / \partial x_1 \quad (22a)$$

or if

$$A_1 \delta (-\overline{u_1 u_1}) / \partial x_1 \ll -\overline{u_1 u_1}. \quad (22b)$$

Now there is a class of flows for which these conditions are not satisfied, and yet there is no demonstrable memory effect. These are flows which possess the property of self-preservation.

### Self-Preserving Flows

These are flows with a pronounced main flow direction, whose structure maintains its similarity during development or decay in this direction. Many free turbulent flows such as free jets and wake flows show such a similarity in the fully developed state. It is convenient to make a distinction between complete and incomplete similarity. In the case of complete similarity, a length scale and a velocity scale are sufficient to make the thus-reduced velocity distributions coincide completely. In the case of incomplete similarity, more than one velocity scale and/or more than one length scale is required.

To determine the possibility of the existence of a self-

/410

preserving flow, let us set

$$\begin{aligned} \bar{U}_1 &= U \left( \frac{x_1}{L_0} \right)^p f(\eta), & \bar{U}_2 &= U \left( \frac{x_1}{L_0} \right)^q g(\eta), & -\overline{u_1 u_1} &= U^2 \left( \frac{x_1}{L_0} \right)^s h(\eta), \\ \eta &= \frac{x_2/L_0}{(x_1/L_0)^n}, & \epsilon_m &= UL_0 \left( \frac{x_1}{L_0} \right)^n k(\eta) \quad \text{mit} \quad n = q - p + s. \end{aligned} \quad (23)$$

Here  $L_0$  is a characteristic length, and  $U$  a characteristic velocity.

By substitution in the Reynolds equations, it can be shown that complete similarity exists only if

$$q = 1; p = r = s/2.$$

With the relations in (23), it follows from the dynamic equations that a plane free jet and a round free jet may have complete similarity, where  $n = 1/2$  for the plane case, and  $n = 0$  for the round case. On the other hand, the plane and the round wake flows can have only incomplete similarity, but in both cases  $n = 0$ , and so  $\epsilon_m$  is independent of  $x_1$ .

To study memory effects in self-preserving flows, we must also determine the possibility of similarity for the relaxation equation (16). For this we set

$$A_1 = L_0 \left( \frac{x_1}{L_0} \right)^l l(\eta). \quad (24)$$

Inserting relations (25) and (24) into eqn. (16), we find that  $t = 1$  is a necessary condition for similarity. Thus, the memory length scale must vary linearly with  $x_1$ . For a fully developed plane wake flow,  $s = -1$ ,  $q = 1/2$  and  $p = -1/2$ , and so  $n = 0$ .

Hence, the memory effect present can no longer be directly detected, even if  $\Lambda_1 \approx x_1$  and conditions (22a) and (22b) were not satisfied. This follows from eqn. (10), where for this similar flow the effect of the first term can be allowed for with an apparently larger  $\epsilon_m$ .

### 3. Experimental Investigations

Two different flows with pronounced memory effects are considered: the restoration of a strongly disturbed boundary layer and the development region of the wake flow behind a cylinder.

#### Restoration of a Disturbed Boundary Layer

The boundary layer in question is that along a straight glass plate set up in the plane of symmetry of a wind tunnel. The test section of the wind tunnel is 4.5 m long and has a cross-section of  $0.8 \times 0.7 \text{ m}^2$ . In the experiments the external velocity of the boundary layer was kept constant:  $U_0 = 10.5 \text{ m/s}$ . A hemisphere 40 mm in diameter was attached to the glass plate with the midpoint of its flat undersurface 3.65 m from the leading edge of the plate. The hemisphere was thus entirely submerged in the boundary layer. The three-dimensional wake flow of the hemisphere produces a considerable disturbance of the boundary-layer flow, as shown in Fig. 3.

The velocity measurements were performed with a constant-temperature hot-wire anemometer. The heating wire consisted of platinum-plated tungsten wire 5  $\mu\text{m}$  in diameter. The length of the sensing element was 2 mm in most cases, but occasionally 1 mm. The distance between the wire supports was 10 mm. Measurements were made of: the three components of the mean velocity and the turbulent velocity, the turbulent shear stresses, spatial velocity correlations and one-dimensional spectral distributions of the

/411

energy of the axial turbulent component. Wall shear stresses were measured with a Preston tube wherever possible.

Results of these investigations have already been published [22]. Here we shall again present some of the most important results together with results of further investigations. Equations (11) and (16) form the starting point for interpreting the results. It was assumed as a second approximation that only the change of  $\partial \bar{U}_1 / \partial x_2$  in the main flow direction [that is  $\Lambda_1 \partial^2 \bar{U}_1 / \partial x_1 \partial x_2$  in eqn. (11)] can be considered. This corresponds to the concept that fluid particles which contribute to momentum transfer travel only relatively short transverse distances in the  $x_2$ - and  $x_3$ -direction. However, closer investigation has shown that the neglected terms in eqn. (8) are actually not negligibly small. Since the individual contributions have positive and negative values, though, the sum total proves to be small.

An important assumption is now made regarding eqn. (11): that the eddy viscosities  $\epsilon_m$  and  $\epsilon_m^0$  are equal and have a constant value in the important  $x_1$ -direction.

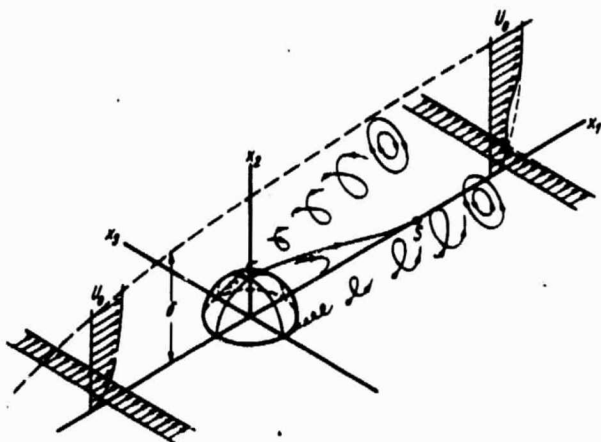


Fig. 3. Wake flow of a hemisphere in a boundary layer.

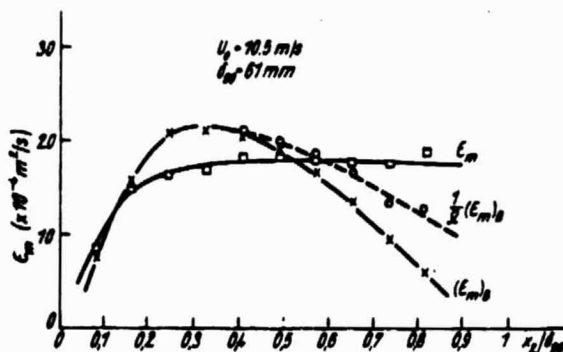


Fig. 4. Distributions of  $(\epsilon_m)_B$ ,  $1/\Omega(\epsilon_m)_B$  and  $\epsilon_m$  acc. to eqn. (11),  $x_1 = 0.50 \text{ m}$ ,  $x_2 = 0.02 \text{ m}$ .

Fig. 4 shows  $\epsilon_m$  as a function of  $x_2/\delta_{99}$ , for  $x_1 = 0.5$  m and  $x_2 = 0.02$  m, first calculated from the Boussinesq relation (first approximation)

$$(\epsilon_m)_B = -\overline{u_2 u_1} / (\partial \bar{U}_1 / \partial x_2),$$

Second,  $\epsilon_m$  is shown corrected for the influence of intermittent turbulence in the outer region of the boundary layer  $1/\Omega(\epsilon_m)_B$ , with  $\Omega$  as the intermittence factor; third, the value is calculated from eqn. (11) and is again corrected for intermittence effects. Lacking our own measurements of  $\Omega$ , we assumed that the values measured by Klebanoff [23] in an undisturbed boundary layer were approximately valid for the disturbed boundary layer as well. It was found that the distribution of  $\epsilon_m$  calculated from eqn. (11), unlike  $(\epsilon_m)_B$ , is similar to the distribution in an undisturbed boundary layer. This is shown in Fig. 5, which also gives the distributions of the dimensionless quantity  $\epsilon_m/u^*\delta_{99}$ , for  $x_1 = 0.25$  m,  $x_2 = 0.02$  m;  $x_1 = 0.50$  m,  $x_2 = 0.02$  m of the disturbed boundary layer and for the same  $x_1$  distances, but  $x_2 = -0.15$  m of the undisturbed boundary layer;  $u^*$  is the shear stress velocity. All values have been corrected with the intermittence factor  $\Omega$ , i.e.,  $(-\overline{u_2 u_1})/\Omega$  was used in eqn. (11). The length  $\Lambda_1$  was calculated from the equation

$$\Lambda_1 = 0.4 \frac{\bar{U}_1}{u_1^*} \Lambda_{1f}, \quad (25)$$

The value 0.4 of the constant is a mean value of measured  $\beta = \Lambda_{1L}/\Lambda_{1f}$  and agrees with the theoretical values found by Saffman [24]. In addition,  $\Lambda_{1f}$  was set equal to  $0.4 \delta_{99}$ , in Fig. 4 and 5.

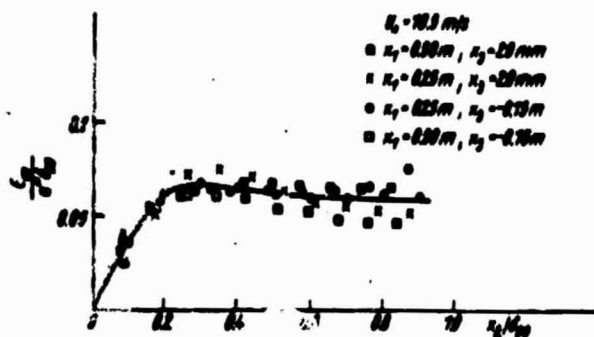


Fig. 5. Distributions of the dimensionless quantity  $\epsilon_m/u^* \cdot \delta_{99}$ ;  $\bullet$  — calculated from  $(-u_2 u_1)$  measurements of Klebanoff [23].

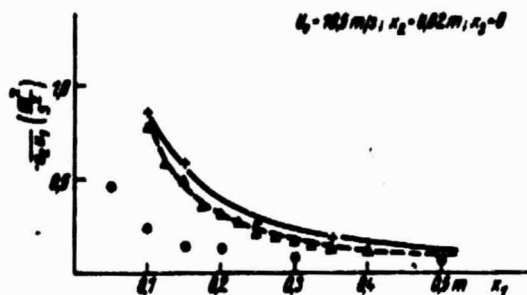


Fig. 6. Turbulent shear stress, measured (+) and calculated by Boussinesq relation (o) and by eqn. (20) (Δ). (Boussinesq:  $\epsilon_m = 0.065 u^* \delta_{99}$ ; eqn. (20):  $\epsilon_m = \langle \epsilon_m \rangle = 14.2 \cdot 10^{-4} \text{ m}^2/\text{s}$ ).

According to our own measurements of the one-dimensional spectral energy distribution of  $u_1$ , i.e. of  $E_1(k_1)$  (where  $k_1$  is the wavenumber in the  $x_1$ -direction), a value of 0.6 appears to be a better approximation than 0.4. The locally measured value of the disturbed boundary layer was taken as  $u^*$ . Because the quotient  $\partial^2 \bar{u}_1 / \partial x_1 \partial x_2$  was not constant along a distance equal to  $\Lambda_1$ , a weighted mean was assumed. It was also found that  $\Lambda_1$  was not constant along a distance equal to  $\Lambda_1$ , as was predicted by eqn. (25). Thus, eqn. (16) [or eqn. (20)] is more suitable if one wishes to consider a variable  $\Lambda_1$ . From measurements of  $\bar{u}_1$ ,  $u'_1$  and  $\Lambda_{1F}$  follows the relation

$$\Lambda_1 = 0.27 x_1^{1/2} \quad (\Lambda_1 \text{ and } x_1 \text{ in m}). \quad (26)$$

Because the result of calculations with various relations for  $\Lambda_1$  has proved to be largely insensitive to changes in this relation, the approximate relation shown above is suitable.

The solution of eqn. (16) with  $\Lambda_1$  according to eqn. (26) is given by eqn. (20) with  $a = 0.27$  and  $n = 0.7$ . Also in eqn. (20),  $x_{10} = 0$  and  $-u_2 u_1(0) = 0.107 \text{ m}^2/\text{s}^2$ . The calculation was performed

along a line  $x = 0.02 \text{ m}$ ,  $x = 0$  (plane of symmetry). Actually the calculation should be carried out along a streamline. But because the difference between a streamline and a line  $x_2 = \text{const}$  is quite small, the calculation was performed with  $x_2 = \text{const}$  for simplicity. According to Fig. 5,  $\epsilon_m/u^*\delta_{,,}$  is independent of  $x_1$ . In eqn. (20) the variable value of  $u^*\delta_{,,}$  should be used for  $\epsilon_m$ . It was found, however, that for  $0.1 \text{ m} < x_1 < 0.6 \text{ m}$ ,  $u^*\delta_{,,}$  and thus  $\epsilon_m$  changes only very little, on the order of  $\epsilon_m \propto x_1^{0.15}$ . Again, for simplicity, a constant mean value  $\epsilon_m = 14.2 \cdot 10^{-4} \text{ m}^2/\text{s}$  was used for the calculation by eqn. (20). Fig. 6 shows the result of the calculation of  $(-\overline{u_2 u_1})$  together with the measured values. The turbulent shear stress was also calculated by the Boussinesq formula, with the local value  $(\epsilon_m)_\beta = 0.065 u^*\delta_{,,}$ .

Although the shear stress  $(-\overline{u_2 u_1})$  calculated by eqn. (20) is still somewhat smaller than the measured shear stress, eqn. (20) shows considerable improvement over calculation by the Boussinesq relation. The shear stress calculated with the local velocity gradient  $\partial \overline{u_1}/\partial x_2$  is much too small, particularly in the first region of the wake flow of the hemisphere. For large  $x_1$ , at which the disturbed boundary layer is nearly restored, the two calculated shear stresses approach the measured value.

It may be asked whether memory effects in the sense understood here might not be present in the undisturbed boundary layer. Relation (26) does not satisfy condition (24) with  $t = 1$  for similarity. Hence, memory effects must be negligibly small. This means that conditions (22a) and (22b) must be satisfied. With  $A_1$  calculated according to eqn. (25), we find for the undisturbed boundary layer

$$\left| \frac{A_1}{-\overline{u_1 u_1}} \frac{\partial}{\partial x_1} (-\overline{u_1 u_1}) \right| \approx 0(0.01),$$



so that condition (22b) may be satisfied with certainty.

### Wake Flow of a Cylinder

This investigation was conducted in a wind tunnel similar to that described above. The measurements were made with cylinders of various diameters. Here we shall consider only the results obtained for a cylinder of  $D = 0.04$  m. With the velocity of the undisturbed wind tunnel flow  $U_0 = 10.5$  m/s, the Reynolds number  $Re_D = U_0/D = 26,000$ .

The "classical" investigations of Townsend [8] in a cylinder wake (with  $Re_D = 1360$ ) already showed that self-preservation of the mean flow velocity occurs no sooner than  $x_1/D = 70$  to 100, because even greater downstream distances are necessary for turbulence. The investigations performed here pertain to the region  $2 < x_1/D < 90$ ; the region  $20 < x_1/D < 50$  was of particular interest in terms of the intended memory effects. For  $x_1/D < 20$ , effects of more or less discrete Kármán eddies are noticeable, depending on the Reynolds number.

The hot-wire anemometer described above was used to measure the mean velocity components  $\bar{U}_1$  and  $\bar{U}_2$ , the turbulence intensities  $u'_1$  and  $u'_2$ , the shear stress  $(-\overline{u_2 u_1})$  and spectral distributions of  $u_1$ . From the measured values of  $\bar{U}_1$ ,  $u'_1$  and the spectral distributions,  $\Lambda_1(x_1, x_2)$  was calculated.

For  $x_2 = \text{const}$ ,  $\Lambda_1 = \text{const } x_1$ , and thus  $\Lambda_1$  is a linear function of  $x_1$ . The eddy viscosity  $\epsilon_m$  was computed for the cross-section  $x_1 = 22.5 D$  as a function of  $x_2/(x_2)_{99}$ . Here  $(x_2)_{99}$  is the lateral distance for which the velocity difference with respect to the free stream is one percent of the maximum local velocity difference. The calculation was performed only for the inner region, in which the turbulence still shows practically no

intermittence. Fig. 7 shows  $\epsilon_m/U_0 D$  according to Boussinesq, according to eqn. (11) with  $\epsilon_m = \epsilon_m^0$  and according to eqn. (16). The equilibrium value for larger  $x_1$ , at which the flow is fully developed is equal to 0.018. A memory effect is clearly evident. The values calculated by eqn. (11) and eqn. (16) are almost equal to this equilibrium value. This value is somewhat higher than 0.0164, the value that follows from the measurements of Townsend [20] with  $Re_D = 1360$ .

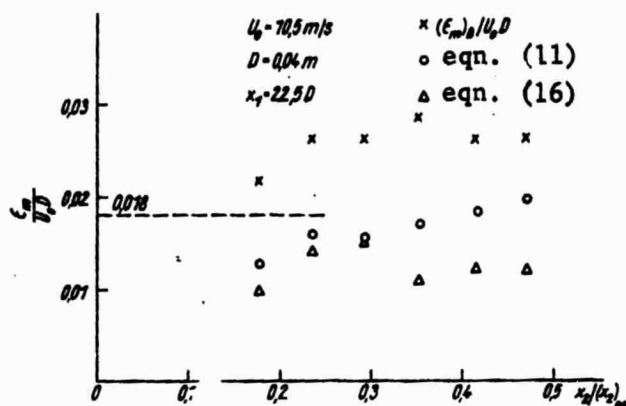


Fig. 7. Distribution of  $\epsilon_m/U_0 D$  according to Boussinesq (x), eqn. (11) (o) and eqn. (16) ( $\Delta$ ).

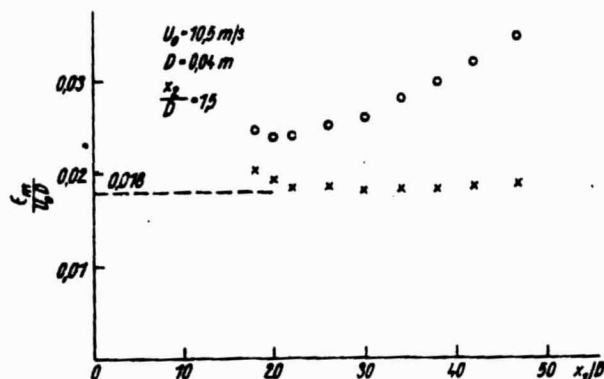


Fig. 8. Distribution of  $\epsilon_m/U_0 D$  in a cylinder wake along the line  $x_2/D = 1.5$ ; calculated according to Boussinesq (o) and eqn. (11) (x).

The variation of  $\epsilon_m/U_0 D$  in an axial direction was calculated according to eqn. (11) along a line  $x_2/D = 1.5$ . This line passes through the point in cross-section  $x_1 = 18 D$  at which the shear stress attains a maximum. In eqn. (11)  $\Lambda_1$  was calculated from eqn. (25), where  $\Lambda_{1f} = 0.5$  to  $0.6 (x_2)_g$ , according to the spectral measurements for  $x_2 = 1.5 D$ . This leads to  $\Lambda_1 \approx 0.3$  to  $0.36 x_1$ , and thus the linear relation previously mentioned. Along the line  $x_2 = 1.5 D$ , the intermittence factor  $\Omega = 1$ . Fig. 8 shows  $\epsilon_m/U_0 D$  calculated according to Boussinesq and according to eqn. (11). After  $x_1/D = 20$  the value calculated from this equation is

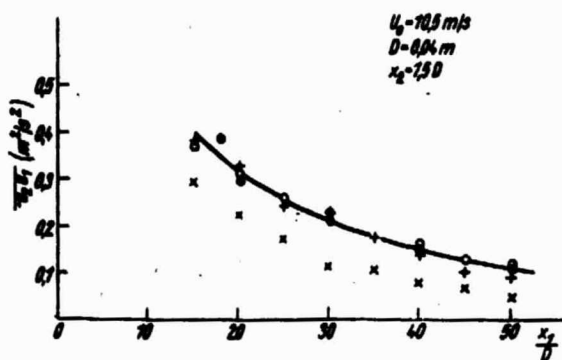


Fig. 9. Axial shear stress distribution in a cylinder wake, measured (●) and calculated according to Boussinesq (x), eqn. (19) (○) and eqn. (11) (+).

practically constant and equal to the equilibrium value of 0.018. The memory effect according to eqn. (19) was investigated along the same line  $x_2 = 1.5 D$ . Here, according to Fig. 8,  $\epsilon_m = \text{const}$  along this line. At point  $x_{10} = 0$ , it was assumed that  $-\overline{u_2 u_1}(0) = 0$ . The result of the calculation with  $\epsilon_m = 0.018 U_0 D = 0.0076 \text{ m}^2/\text{s}$  and  $a = 0.3$  is shown in Fig. 9, together with  $(-\overline{u_2 u_1})$  calculated according to Boussinesq with the same value of  $\epsilon_m$ . This figure also shows  $(-\overline{u_2 u_1})$  calculated according to eqn. (11) and the

measured variation of  $(-\overline{u_2 u_1})$ .

The linear relation  $\Lambda_1 \simeq 0.3 x_1$  can only be an approximation. In reality  $\Lambda_1$  must increase somewhat parabolically with  $x_1$  before it can vary linearly with  $x_1$  in the fully-developed equilibrium region in which similarity obtains. According to eqn. (25),  $\Lambda_1 \propto \Lambda_{1f} U_0 / u_1'$ , because  $\overline{U_1} \simeq U_0$  for very large  $x_1/D$ . In the similarity region,  $\Lambda_{1f} \propto (x_1 D)^{1/2}$  and  $u_1' / U_0 \propto (D/x_1)^{1/2}$ ; thus,  $\Lambda_1 \propto x_1$ . From the measurements of Townsend [8] it can be calculated that  $\Lambda_1 \simeq 0.26 x_1$ .

With a linear relation for  $\Lambda_1$ , eqn. (16) can satisfy the conditions for similarity. The following values for the exponents in relation (23) have been obtained for self-preserving wake flow:  $p = -1/2$ ,  $q = 1/2$  and  $s = -1$ ; thus,  $n = 0$ , and  $\epsilon_m$  is independent of  $x_1$ . As stated earlier, a memory effect is no longer directly detectable under these circumstances. To determine the possibility of its existence, one must weigh the quantities of the terms in

eqn. (16) against each other. A rough estimate based on the measurements of Townsend yields

$$\left| \frac{A_1}{-u_1 u_1} \frac{\partial}{\partial x_1} (-\overline{u_1 u_1}) \right| \approx 0,2 \text{ bis } 0,3,$$

It is apparent that this term is not small. It is possible, then, that the memory effect in question is present in the entire wake flow, but is not directly detectable due to the similarity of the flow. In this case a theoretical solution with the Boussinesq relation for shear stress and an adjusted value of  $\epsilon_m$  without a memory effect may be satisfactory.

#### 4. Concluding Remarks

The above theoretical considerations, supported by experimental findings, have clearly shown that the additional memory effects in turbulence can play an essential role, especially if we are dealing with flows which fluctuate relatively rapidly in the flow direction, so that the turbulence has no opportunity to adapt completely to the local state of the mean flow. Then the Reynolds stresses, for example, can be thought of as consisting of two parts. The first part is practically "local" in character and is determined by the local state of the flow, while the second part is an additional contribution from the flow's prior history, i.e., an additional memory effect due to a relatively rapid change in the flow state. These additional memory effects can still be present in fully developed turbulent flows which are practically in equilibrium. If the flow is self-preserving, however, these memory effects are not directly detectable. The above investigations suggest that memory effects are always present in self-preserving wake flows, because these effects are negligibly small in a fully developed turbulent wall flow (boundary layer).

When we consider relations which are local in character, such as the Boussinesq relation, as a first approximation, the above expansions are nothing more than a second approximation. This point must be reemphasized.

As stated earlier, memory effects are directly detectable in the development region of a turbulent flow, and not only in a wake flow, but also in a free jet. An investigation of memory effects in the development region of a grid flow is currently in progress. It is known that the turbulence of the fully developed grid flow is axially symmetrical with  $\overline{u_1^2} > \overline{u_2^2} = \overline{u_3^2}$ . If these quantities are interpreted as normal stresses, the normal stress in the axial direction  $\overline{u_1^2}$  is greater than the normal stresses in the transverse direction. Thus, the shear stress must be maximum in areas which form an angle  $\pi/4$  with the axial direction. This is incompatible with the absence of a mean velocity gradient, if we assume a simple relation between shear stress and deformation rate. The mean flow and turbulence are not yet homogenous in the development region, and memory effects can produce the aforementioned anomaly, which may exist for an unlimited distance downstream provided the memory effects are sustained.

One difficulty in making a quantitative determination of memory effects is our lack of knowledge with respect to relaxation time, memory function and Lagrangian autocorrelation. Relation (25) is nothing more than an approximation. Moreover, the determination of the axial derivative of  $\partial \overline{U_1} / \partial x_2$  in eqn. (11) and of  $\partial / \partial x_1 (-\overline{u_2 u_1})$  in eqn. (16) can be performed only with a low degree of accuracy. The large scatter of the points, as in Fig. 5 and 7, is due largely to this fact. In this respect a quantitative determination of the memory effect by means of eqn. (17), which requires integration instead of differentiation, offers significant advantages.

Our considerations here have been limited to turbulent momentum transfer. Naturally, such memory effects also play a role in the transport of other transferrable quantities such as heat and mass.

## REFERENCES

1. Lumley, J. L., J. Fluid Mech. 41, 413 (1970).
2. Clauser, F. H., Advances in Appl. Mech. 4, 1 (1956).
3. Landahl, M. T., J. Fluid Mech. 29, 441 (1967).
4. Lissenburg, R. C. D., J. O. Hinze and H. Leijdens, Appl. Sci. Res. 31, 343 (1975).
5. Philip, J. R., Phys. Fluids 11, 38 (1968).
6. Corrsin, S., Advances in Geophysics 18 A, 25 (1974).
7. Prandtl, J., Z. angew. Math. Mech. 22, 241 (1942).
8. Townsend, A. A., Australian J. Sci. Res. 2 A, 451 (1949).
9. Eskinazi, S. and H. Yeh, J. Aeronaut. Sci. 23, 23 (1956).
10. Tailland, A. and J. Mathieu, J. Mécanique 6, 103 (1967).
11. Kjellström, B. and S. Hedberg, Report A.E.-243, Aktiebolaget Atomenergie, Stockholm, 1966.
12. Bégulier, C., Etude du jet plan dissymétrique en régime turbulent incompressible [Study of a Plane Asymmetrical Jet in an Incompressible Turbulent Regime], Dissertation, University of Aix-Marseille, 1971.
13. Hanjalic, K. and B. E. Launder, J. Fluid Mech. 52, 609 (1972).
14. Hinze, J. O., Appl. Sci. Res. 22, 163 (1970).
15. Eskinazi, S. and F. F. Erian, Phys. Fluids 12, 1988 (1969).
16. Phillips, O. M., J. Fluid Mech. 27, 131 (1967).
17. Deissler, R. G., J. Fluid Mech. 64, 763 (1974).
18. Crow, S. C., J. Fluid Mech. 33, 1 (1968).
19. Amini, J. A. (personal communication).
20. Hinze, J. O., Turbulence, Chap. 4, 2nd edition, McGraw-Hill Book Co., New York, 1975.
21. Rotta, J., Z. f. Physik 129, 547 (1951).
22. Hinze, J. O., R. E. Sonnenberg and P. J. H. Builtjes, Appl. Sci. Res. 29, 1 (1974).

23. Klebanoff, P. S., N.A.C.A. Tech. Notes, No. 3178 (1954).
24. Saffman, P. G., Appl. Sci. Res. 11 A, 245 (1963).
25. Philip, J. R., Phys. Fluids 10, Suppl. 9, 69 (1967).
26. Antonia, R. A. and R. E. Luxton, J. Fluid Mech. 48, 721 (1971).